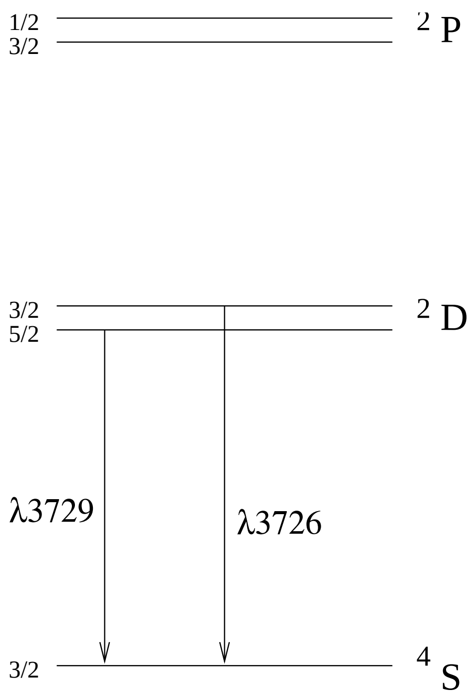

The energy level diagram of singly ionized oxygen is displayed on the left. The wavelengths, transition probabilities, and collision strengths are as follows

Transition	Wavelength	A (sec^{-1})	Ω
$^2P_{1/2} \rightarrow ^2P_{3/2}$	4.88 mm	1.4×10^{-10}	0.29
$^2P_{1/2} \rightarrow ^2D_{3/2}$	7329.6 Å	9.4×10^{-2}	0.28
$^2P_{1/2} \rightarrow ^2D_{5/2}$	7318.8 Å	5.6×10^{-2}	0.30
$^2P_{1/2} \rightarrow ^4S_{3/2}$	2470.2 Å	2.4×10^{-2}	0.133
$^2P_{3/2} \rightarrow ^2D_{3/2}$	7330.7 Å	5.8×10^{-2}	0.41
$^2P_{3/2} \rightarrow ^2D_{5/2}$	7319.9 Å	1.1×10^{-1}	0.73
$^2P_{3/2} \rightarrow ^4S_{3/2}$	2470.3 Å	5.8×10^{-2}	0.267
$^2D_{3/2} \rightarrow ^2D_{5/2}$	496 μ	1.3×10^{-7}	1.17
$^2D_{3/2} \rightarrow ^4S_{3/2}$	3726.0 Å	1.8×10^{-4}	0.536
$^2D_{5/2} \rightarrow ^4S_{3/2}$	3728.9 Å	3.6×10^{-5}	0.804

Other constants you may need are given below (in cm^3 s^{-1})

Value (cm^3 s^{-1})	5000 K	10,000 K	20,000 K
α_B	4.54×10^{-13}	2.59×10^{-13}	1.43×10^{-13}
β_B	3.22×10^{-13}	1.73×10^{-13}	9.17×10^{-14}
$\alpha^{eff}_{\mathrm{H}\beta}$	5.38×10^{-14}	3.02×10^{-14}	1.61×10^{-14}

Now consider a very peculiar $\log(L/L_\odot) = 4.5$ star that emits all its energy monochromatically at 700 Å. Surrounding this star is an almost pure hydrogen nebula, contaminated only by trace amounts of oxygen. The abundance of oxygen (by number) relative to hydrogen, is 5×10^{-4}. The nebula is uniform and homogeneous, with a constant density of $N_e = 100$ cm^{-3}. This density is low enough so that you may neglect all collisional de-excitation.

1) What ionization state will the oxygen be in?

The ionization potential for neutral oxygen is 13.618 eV, which is equivalent to photons of 910 Å. It takes 35.11 eV, or 353 Å light to strip a second electron off oxygen. The monochromatic star can therefore create singly ionized oxygen, but not doubly ionized oxygen.

2) At $T \approx 10,000$ K, what is the critical density where collisions out of the ${}^2D_{5/2}$ state become more important than radiative transitions out of the state? (In other words, is the assumption that you can neglect collisional de-excitations a good one?)

The rate at which radiative transitions occur out of the ${}^2D_{5/2}$ state is $A = 3.6 \times 10^{-5}$ sec^{-1}. Collisions become more important than radiative transitions when

$$N_e \sum_{j \neq i} q_{ij} > \sum_{j<i} A_{ij}$$

where

$$q_{ij} = 8.629 \times 10^{-6} \frac{\Omega_{i,j}}{\omega_i T^{1/2}} e^{-\Delta E/kT} \text{ cm}^3 \text{ s}^{-1}$$

and $\Delta E = 0$, if a collisional de-excitation is being considered. For transitions out of ${}^2D_{5/2}$, $J = 5/2$, so $\omega_i = (2J + 1) = 6$.

Plugging in the numbers, yields

$$\begin{aligned}
q &= 6.04 \times 10^{-10} \text{ cm}^3 \text{ s}^{-1} \quad \text{for} \quad {}^2D_{5/2} \rightarrow {}^2P_{1/2} \\
q &= 1.47 \times 10^{-9} \text{ cm}^3 \text{ s}^{-1} \quad \text{for} \quad {}^2D_{5/2} \rightarrow {}^2P_{3/2} \\
q &= 1.68 \times 10^{-8} \text{ cm}^3 \text{ s}^{-1} \quad \text{for} \quad {}^2D_{5/2} \rightarrow {}^2D_{3/2} \\
q &= 1.16 \times 10^{-8} \text{ cm}^3 \text{ s}^{-1} \quad \text{for} \quad {}^2D_{5/2} \rightarrow {}^4S_{3/2}
\end{aligned}$$

Thus, for collisions to be important,

$$N_e > \frac{\sum_{j<i} A_{ij}}{\sum_{j \neq i} q_{ij}} = 1200 \text{ cm}^{-3}$$

So collisional de-excitations are not important.

3) Assume the on-the-spot approximation and calculate the temperature of the free electrons in the nebula. (You will have to do some interpolation here.)

If all the ionizing photons are absorbed, then under the on-the-spot approximation, the energy input to the nebula is

$$G(H) = N_e N_p \alpha_B(T) \cdot \frac{3}{2} k T_i$$

where the initial temperature, T_i is given by

$$\frac{3}{2} k T_i = \frac{\int_{\nu_0}^{\infty} \frac{J_\nu}{h\nu} h(\nu - \nu_0) d\nu}{\int_{\nu_0}^{\infty} \frac{J_\nu}{h\nu} d\nu}$$

For a monochromatic star, the integrals become trivial, and

$$\frac{3}{2} k T_i = \frac{hc}{\lambda} - \frac{hc}{\lambda_0}$$

where $\lambda = 700$ Å, and $\lambda_0 = 912$ Å. Since $N_e \approx N_p = 100$ cm^{-3}, the initial temperature is $T_i = 31,850$ K, and the energy input is

$$G(H) = N_e N_p \alpha_B(T) \cdot \frac{3}{2} k T_i = 6.59 \times 10^{-8} \ \alpha_B(T) \text{ ergs cm}^{-3} \text{ s}^{-1}$$

The energy lost by free-free emission is given by

$$L_{ff} = 1.42 \times 10^{-27} Z^2 T^{1/2} g_{ff} N_e N_+$$

The two constituents of the nebula are hydrogen ($Z = 1, N_+ = 0.9995 N_e$) and oxygen ($Z = 8, N_+ = 0.0005 N_e$). So

$$L_{ff} = 1.42 \times 10^{-23} \left[0.9995 + (0.0005 * 64)\right] T_e^{1/2} = 1.46 \times 10^{-23} \, T_e^{1/2} \text{ ergs cm}^{-3} \text{ s}^{-1}$$

Meanwhile, the energy lost by recombination (in a pure hydrogen nebula) is

$$L_R = N_e N_p k T_e \beta_B(T) = 1.38 \times 10^{-12} T_e \beta_B(T) \text{ ergs cm}^{-3} \text{ s}^{-1}$$

For collisional cooling of singly ionized oxygen in the low density limit, all collisions occur from the ground state, and every collision upward is followed by a radiative transition downward. Thus, the cooling is

$$L_C = N_e N_i \, q_{ij} h\nu_{ij} = N_e N_I h\nu_{ij} \cdot 8.629 \times 10^{-6} \frac{\Omega_{i,j}}{\omega_i T^{1/2}} e^{-\Delta E/kT}$$

where ω_i, the statistical weight of the ground state is 4. For $N_e = 100$ and $N_i = 5 \times 10^{-4} N(H) = 0.05$ the total cooling rate is then

$$L_C = 2.14 \times 10^{-13} T^{-1/2} \sum_{j=2,5} \frac{\Omega_{1,j}}{\lambda_{1,j}} e^{hc/\lambda_{1,j} kT}$$

For the nebula to be in thermal balance, $G(H) = L_{ff} + L_R + L_C$. If you interpolate over α_B and β_B, you find the equilibrium temperature of the nebula to be around $T \approx 9500$ K.

4) Recalculate the temperature assuming the abundance of O^+ is identically zero.

Without the collisional cooling, the equilibrium temperature is $T \approx 33,200$ K. Note that this is hotter than the initial temperature.

5) What is the size of the nebula?

The size of the nebula is given by the Strömgren sphere equation,

$$Q(H) = \frac{4}{3}\pi\mathcal{R}^3 N(H)^2 \alpha_B$$

For a monochromatic star with $\lambda = 700$ Å and luminosity $\log L/L_\odot = 4.5$ ($L = 1.2 \times 10^{38}$ ergs s^{-1}), the number of ionizing photons is simply

$$Q(H) = \frac{L}{h\nu} = 4.25 \times 10^{48} \text{ photons s}^{-1}$$

A linear interpolation gives $\alpha_B \approx 2.82 \times 10^{-13}$ cm^3 s^{-1} at $T \approx 9500$ K. Plugging this in gives, $\mathcal{R} = 2.3$ pc.

6) What is the total Hβ luminosity from the nebula? How much luminosity would you expect from Hα?

The value $\alpha^{eff}_{\mathrm{H}\beta}$ is the effective recombination coefficient for Hβ (λ4861 Å), i.e., the number of hydrogen recombinations that eventually make the $4 \rightarrow 2$ transition. Thus the number of Hβ photons produced is

$$N(H\beta) = N_e N(H) \alpha^{eff}_{\mathrm{H}\beta}$$

and the Hβ emission is

$$N_e N(H) \alpha^{eff}_{\mathrm{H}\beta} h\nu_{\mathrm{H}\beta}$$

Interpolating at $T = 9500$ gives $\alpha^{eff}_{\mathrm{H}\beta} = 3.30 \times 10^{-14}$ cm^3 s^{-1}. So the emission at Hβ is 1.34×10^{-21} cm^{-3} s^{-1}. When you multiply this over the entire volume of the Strömgren sphere, you get 2.0×10^{36} ergs s^{-1}, or 530 $L/L_\odot$ of Hβ emission. Since for Case-B recombination, Hα is roughly 2.86 times that of Hβ, you get an Hα luminosity of 5.7×10^{36} ergs s^{-1}, or 1490 $L/L_\odot$.

7) Compute the strengths of all the emission lines produced by O^+ relative to Hβ. What is the brightest line in the nebula?

The number of ions in each level of oxygen is given by the collisional excitation rate up to that level, i.e.,

$$n_j = N_e N_I \, q_{i,j} = N_e N_I \, 8.629 \times 10^{-6} \frac{\Omega_{1,j}}{\omega_1 T^{1/2}} \, e^{hc/\lambda_{1,j} kT} \text{ cm}^{-3} \text{ s}^{-1}$$

Each of these excitations will create a photon with energy $h\nu_{ij}$. Thus the number of electrons in the ${}^2P_{1/2}$ state (level 5) will be

$$n_5 = N_e N_i \, q_{5,1}$$

and the strengths of the lines coming from this level will be

$$I_{5,i} = n_5 \cdot \frac{A_{5,i}}{\sum\limits_{i<5} A_{5,i}} \cdot \frac{hc}{\lambda_{5,i}}$$

where the second term gives the fraction of decays into each level. This equation also yields a good approximation for the line strengths arising from all the other states as well. However, if you want to be precise, you must also consider the contribution of electrons that have arrived in the lower states from decays from the upper states. In other words, the number of electrons in the ${}^2P_{3/2}$ (level 4) state will be

$$n_4 = N_e N_i \, q_{4,1} + n_5 \frac{A_{5,4}}{\sum\limits_{i<5} A_{5,i}}$$

and the number of electrons in the ${}^2D_{3/2}$ (level 3) state will be

$$n_3 = N_e N_i \, q_{3,1} + n_5 \frac{A_{5,3}}{\sum\limits_{i<5} A_{5,i}} + n_4 \frac{A_{4,3}}{\sum\limits_{i<4} A_{4,i}}$$

and so on.

Meanwhile, the energy energy emitted at Hβ per unit volume per second is

$$N_e N(H) \alpha^{eff}_{\mathrm{H}\beta} \, h\nu_{\mathrm{H}\beta}$$

So the ratio of collisional excited transition from level i to level j to Hβ will be

$$R = \frac{n_i \frac{A_{i,j}}{\sum\limits_{j<i} A_{i,j}} h\nu_{ij}}{N_e N(H) \alpha^{eff}_{\mathrm{H}\beta} \, h\nu_{\mathrm{H}\beta}}$$

Plugging in the numbers gives

Transition	Wavelength (Å)	Ratio to Hβ
${}^2D_{5/2} \longrightarrow {}^4S_{3/2}$	3729	6.50
${}^2D_{3/2} \longrightarrow {}^4S_{3/2}$	3724	4.34
${}^2D_{3/2} \longrightarrow {}^4D_{5/2}$	496000	2.3×10^{-5}
${}^2P_{3/2} \longrightarrow {}^4S_{3/2}$	2470	0.103
${}^2P_{3/2} \longrightarrow {}^4D_{5/2}$	7320	0.066
${}^2P_{3/2} \longrightarrow {}^4D_{3/2}$	7331	0.034
${}^2P_{1/2} \longrightarrow {}^4S_{3/2}$	2470	0.027
${}^2P_{1/2} \longrightarrow {}^4D_{5/2}$	7319	0.022
${}^2P_{1/2} \longrightarrow {}^4D_{3/2}$	7330	0.036
${}^2P_{1/2} \longrightarrow {}^4P_{3/2}$	4.88×10^7	8.13×10^{-15}

[O II] λ3729 is the brightest line in the nebula.